$$\Rightarrow \qquad \lim_{n\to\infty}\left(\int_0^1 \sqrt[n]{x^n+\alpha}\,\mathrm{dx}\right)^{\mathrm{n}}$$

$$= \lim_{n\to\infty}\left((c^n+\alpha)^{\frac{1}{n}}\right)^n = \lim_{n\to\infty}(c^n+\alpha) = \alpha$$

since $c \in (0,1)$ and $c^n \overset{n\to+\infty}{\longrightarrow} 0$.

Also solved by Kee-Wai Lau, Hong Kong, China; Carl Libis (two solutions; one alone and one with Tom Dunion), Ivy Bridge College of Tiffin University, Toledo, OH and Bentley University, Waltham, MA (respectively); Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy, and the proposer.

Mea Culpa

The names of Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany were inadvertently not listed as having solved problem 5232.

The featured solutions to Problem 5229 have turned out to be in error, or perhaps more correctly stated, incomplete. Following is a note received from **Arkady Alt of San Jose, CA.**

I'm writing you about problem 5229. I think that there are some issues with the proposed solutions and I wanted to give a few arguments to prove this point. Also, below, I'm attaching my solution that I have not posted after realizing that it is not complete, although I did obtain the desired limit.

There are two main approaches to finding limits. Both are in two steps.

The first way is to prove that limit exists and then find it;

The second way is to find the value of the limit assuming that it exists, and then prove that the obtained value is indeed a limit.

The second way isn't complete without such a proof, because there are counterexamples of sequences which have no limit, but when assuming that it exists we can obtain a value.

For example: let $a_1 = 1$ and $a_{n+1} = a_n^2 + 3a_n + 1, n \geq 1$ then obviously $\lim\limits_{n\to\infty} a_n = \infty$.

But assuming that $(a_n)_{n\geq 1}$ is convergent and denoting $a = \lim\limits_{n\to\infty} a_n$ we immediately obtain

$$a = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\left(a_n^2+3a_n+1\right) = \lim_{n\to\infty} a_n^2 + 3\lim_{n\to\infty} a_n + 1 = a^2+3a+1a^2+2a+1 = 0 \iff a = -1.$$

Also, the Stolz Theorem cannot be inverted.

Example:

Let $a_n = \sum_{k=1}^{n} \sin k$, then $\frac{a_{n+1} - a_n}{n+1-n} = \sin(n+1)$ and the sequence $(\sin n)_{n \in N}$ isn't convergent, but

since $\sum_{k=1}^{n} \sin k = \frac{\sin\left(\frac{n+1}{2}\right)\sin\frac{n}{2}}{\sin\frac{1}{2}}$ ($2a_n \sin\frac{1}{2} = \sum_{k=1}^{n}\left(\cos\left(k-\frac{1}{2}\right) - \cos\left(k+\frac{1}{2}\right)\right) =$
$\cos\frac{1}{2} - \cos\left(n+\frac{1}{2}\right) = 2\sin\left(\frac{n+1}{2}\right)\sin\frac{n}{2}$) then $\lim_{n\to\infty}\frac{a_n}{n} = 0$ because
$\left|\sin\left(\frac{n+1}{2}\right)\sin\frac{n}{2}\right| \leq 1.$

Here is my solution, which I decided not to send because it is missing the crucial "proof" points that are mentioned above and it is only based on an assumption. (Note that the published solutions 2 and 3 for problem 5229 are incomplete for the same reason).

Solution 1 is also incomplete (for another reason) because it is based on an unproved assumption about the asymptotic behavior of $(x_n)_{n\geq 1}$, namely that $x_n \sim kn^{\alpha}$, for some k and α.

This assumption is basically equivalent to the problem statement.

I have a slight suspicion that a "simple" solution from the proposer was originally the rationale for the publication of this problem.

So, in my opinion this problem has not been solved as of yet.

5229. Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.

Let $\beta, a > 0$ be a real numbers and let $\{x_n\}_{n\in N}$ be the sequence defined by the recurrence
relation

$$x_1 = a, \text{ and } x_{n+1} = x_n + \tfrac{n^{2\beta}}{x_1+x_2+\ldots+x_n} \quad \text{for } n \geq 1.$$

1. Prove that $\lim_{n\to\infty} x_n = \infty$;
2. Calculate $\lim_{n\to\infty}\frac{x_n}{n^{\beta}}$.

Solution by Arkady Alt, San Jose ,CA

1. Let $S_n := x_1 + x_2 + \ldots + x_n, n \in N$. It is easy to see (by Math. Induction) that $x_n > 0$ for all $n \in N$.

Also, note that sequence $\{x_n\}_{n\in N}$ is increasing, since
$x_{n+1} - x_n = \frac{n^{2\beta}}{S_n} > 0 \iff x_{n+1} > x_n$, $n \in N$.

Then $x_{n+1}^2 - x_n^2 = \frac{n^{2\beta}\left(x_n + x_{n+1}\right)}{S_n} > \frac{2n^{2\beta}x_n}{nx_n} = 2n^{2\beta-1}, n \in N$ and, therefore,
$x_{n+1}^2 - x_1^2 = \sum_{k=1}^{n}\left(x_{k+1}^2 - x_k^2\right) > 2\sum_{k=1}^{n} k^{2\beta-1} > \frac{n^{2\beta}}{\beta}x_{n+1}^2 > a + \frac{n^{2\beta}}{\beta} > \frac{n^{2\beta}}{\beta}x_n > \frac{(n-1)^{\beta}}{\sqrt{\beta}}.$
Thus, $\lim_{n\to\infty} x_n = \infty.$

We can prove that sequence $\frac{x_n}{n^\beta}$ has an upper bound.

Indeed, since $x_n > \frac{(n-1)^\beta}{\sqrt{\beta}}$ then $S_n > \sum_{k=1}^{n} \frac{(k-1)^\beta}{\sqrt{\beta}} > \frac{1}{\sqrt{\beta}} \sum_{k=1}^{n-1} k^\beta > \frac{(n-1)^{\beta+1}}{(\beta+1)\sqrt{\beta}}$ and, therefore ,

$x_{n+1} - x_n = \frac{n^{2\beta}}{S_n} < \frac{n^{2\beta}(\beta+1)\sqrt{\beta}}{(n-1)^{\beta+1}} = n^{\beta-1} \cdot \left(1 + \frac{1}{n-1}\right)^{\beta+1} (\beta+1)\sqrt{\beta} < Kn^{\beta-1}$,

where

$K = e(\beta+1)\sqrt{\beta}$, because $\left(1 + \frac{1}{n-1}\right)^{\beta+1} < \left(1 + \frac{1}{n-1}\right)^{n-1} < e$ for any n bigger then some $n_0 > 0$.

Then $x_{n+1} - x_{n_0} < \frac{K(n+1)^\beta}{\beta} \frac{x_n}{n^\beta} < \frac{x_{n_0}}{n^\beta} + \frac{K}{\beta}, n \geq n_0$.

If I can prove that $\frac{x_n}{n^\beta}$ is increasing, then we can conclude that $\lim_{n\to\infty} \frac{x_n}{n^\beta}$ exists. Attempts to do so failed.

Or, assuming that $\left(\frac{x_n}{n^\beta}\right)_{n\in N}$ is convergent we can try to find $L = \lim_{n\to\infty} \frac{x_n}{n^\beta}$, but later we must prove that the obtained value is really the desired limit. Value of L can be obtained repeatedly using Stolz Theorem:

Indeed, using* $\lim_{n\to\infty} \frac{(n+1)^\alpha - n^\alpha}{\alpha n^{\alpha-1}} = 1, \alpha > 0$ we obtain

L$= \lim_{n\to\infty} \frac{x_n}{n^\beta} = \lim_{n\to\infty} \frac{x_{n+1} - x_n}{(n+1)^\beta - n^\beta} = \lim_{n\to\infty} \frac{x_{n+1} - x_n}{\beta n^{\beta-1}} = \lim_{n\to\infty} \frac{n^{2\beta}}{\beta n^{\beta-1} S_n} = \frac{1}{\beta} \lim_{n\to\infty} \frac{n^{\beta+1}}{S_n} =$

$\frac{1}{\beta} \lim_{n\to\infty} \frac{(n+1)^{\beta+1} - n^{\beta+1}}{S_{n+1} - S_n} = \frac{1}{\beta} \lim_{n\to\infty} \frac{(n+1)^{\beta+1} - n^{\beta+1}}{S_{n+1} - S_n} = \frac{\beta+1}{\beta} \lim_{n\to\infty} \frac{n^\beta}{x_{n+1}} =$

$\frac{\beta+1}{\beta} \lim_{n\to\infty} \left(\frac{x_n}{x_{n+1}} \cdot \frac{n^\beta}{x_n}\right) = \frac{\beta+1}{\beta} \lim_{n\to\infty} \frac{n^\beta}{x_n} = \frac{\beta+1}{\beta} . \frac{1}{L} L = \sqrt{\frac{\beta+1}{\beta}}$.

(here, the chain of equalities according to Stolz Theorem works from the right to the left).

But attempts to prove that $\lim_{n\to\infty} \frac{x_n}{n^\beta} = \sqrt{\frac{\beta+1}{\beta}}$ failed as well.

(*) By Mean Value Theorem $(n+1)^\alpha - n^\alpha = \alpha c_n^{\alpha-1}$, where $c_n \in (n, n+1)$ and, therefore, $\alpha \min\left\{n^{\alpha-1}, (n+1)^{\alpha-1}\right\} < (n+1)^\alpha - n^\alpha < \alpha \max\left\{n^{\alpha-1}, (n+1)^{\alpha-1}\right\}$.

Hence, $\alpha \min\left\{1, \frac{(n+1)^{\alpha-1}}{n^{\alpha-1}}\right\} < \frac{(n+1)^\alpha - n^\alpha}{n^{\alpha-1}} < \alpha \max\left\{1, \frac{(n+1)^{\alpha-1}}{n^{\alpha-1}}\right\}$.

Editor again: I sent Arkady's comments to Ovidiu (proposer of the problem), and he answered as follows:

"I have read Prof. Alt's comments on problem 5229 and he is right, namely the applicability of the Stolz-Cesaro lemma is valid provided that $\lim \frac{x_{\{n+1\}} - x_n}{(n+1)^\beta - n^\beta}$ exists, which I failed to prove. It seems hard to establish the existence of this limit. It appears that the solution of this problem is incomplete, as Prof. Alt has observed."

Ovidiu went on to say that he had communicated the above to some of his colleagues, but to date, they had not been able to solve, or circumvent the glitch.